Signal and Systems

By

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كلية الهندســـــة بشبرا
FACULTY OF ENGINEERING- SHOUBRA

Lecture (5)

Course Title: **Signal and Systems**

Course Code: **ELE 115**

Contact Hours: **5.**

= [2 Lect. + 2 Tut + 1 Lab]

Assessment:

Final Exam: 75%.

Midterm: ??%.

Year Work & Quizzes: 50%.

Experimental/Oral: 25%.

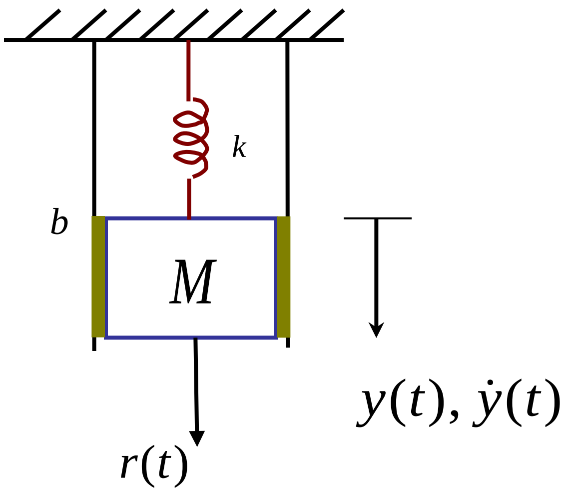
Textbook:

1- E. W. Kamen and B. S. Heck, Fundamentals of Signals and Systems Using the Web and MATLAB, 3rd ed., Pearson Hihgher Education, 2006.
2- Benjamin C. Kuo " Automatic control systems" 9th ed., John Wiley & Sons, Inc. 2010.
3- Katsuhiko Ogata, "Modern Control Engineering", 4th Edition, 2001.

Course Description

- Introduction, fundamentals and basic properties of signals and systems, definition of open loop and closed loop systems, mathematical models of physical systems (mechanical, electrical, electromechanical systems …), control system components, block diagram simplification, signal flow graph, state variable models, Z-Transform and its properties, solving difference equations, pulse transfer function of discrete system, Fourier transforms, continuous and discrete signal analysis, transient response of first and second order control systems, real life applications such as analog and digital filters, introduction to basics of digital signal processor (DSP) and its features and capabilities of commercial applications.

Example



$r: external\ force$

$k: spring$

$b: coefficient\ of\ viscous\ friction$

By Newton's Law

$$F = M\ddot{y}$$

$$r - ky - b\dot{y} = M\ddot{y}$$

$$\Rightarrow M\ddot{y} + b\dot{y} + ky = r$$

let $\boxed{x_1 = y, x_2 = \dot{y}}$

$$\Rightarrow \begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -\dfrac{b}{M}\dot{y} - \dfrac{k}{M}y + \dfrac{1}{M}r \end{cases}$$

$$= -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u \qquad (u = r)$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{M}x_1 - \dfrac{b}{M}x_2 + \dfrac{1}{M}u \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\dfrac{k}{M} & -\dfrac{b}{M} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix}}_{B} \cdot u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

State Space Equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

For example:
$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u \end{cases}$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)}$$

For example: $G(s) = \dfrac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$

Example: Transfer function of the Mass-damper-spring system

$$M\frac{d^2 y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

$$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

$$Ms^2 Y(s) + bsY(s) + kY(s) = U(s)$$

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{Ms^2 + bs + k}$$

Example

MIMO system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|}$$

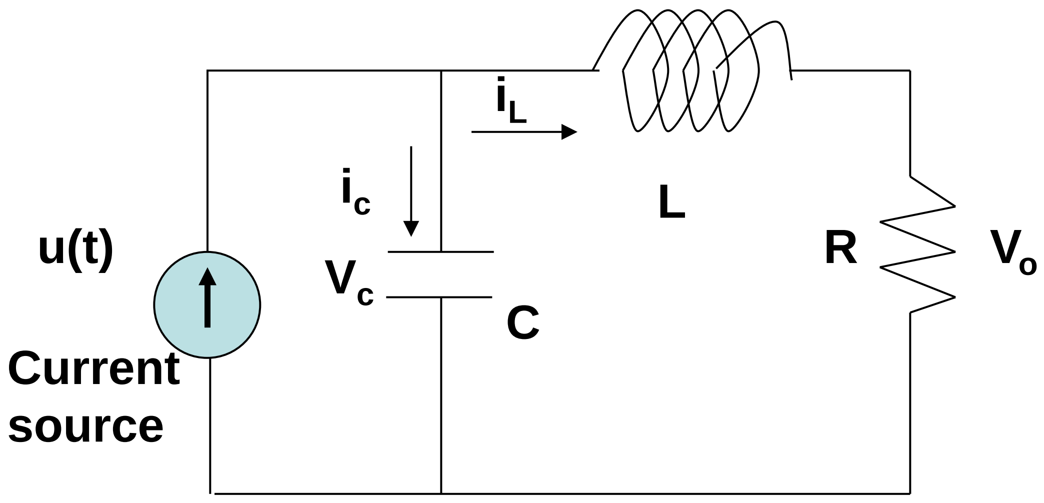
$$= \frac{1}{s(s+4)(s+2)+3+3s} \begin{bmatrix} s^2+6s+11 & s+2 & 3 \\ -3 & s^2+2 & 3s \\ s+4 & -s-1 & s^2+4s \end{bmatrix}$$

$$G(s) = [C(sI - A)^{-1} B + D]$$

$$= \frac{1}{s^3+6s^2+11s+3} \begin{bmatrix} s+2 & 3 \\ -(s+1) & s(s+4) \end{bmatrix}$$

Transfer function

As another example of the state variable characterization of a system, consider the RLC circuit shown in the following figure.



The state of this system can be described in terms of a set of variables [x_1 x_2], where x_1 is the capacitor voltage $v_c(t)$ and x_2 is equal to the inductor current $i_L(t)$. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables.

$$E_1 = \frac{1}{2} L\, i_L{}^2, \quad E_2 = \frac{1}{2C}\left(\int i_c dt\right)^2 = \frac{1}{2} C\, v_c^2$$

Therefore $x_1(t_0)$ and $x_2(t_0)$ represent the total initial energy of the network and thus the state of the system at $t=t_0$.

Utilizing Kirchhoff's current low at the junction, we obtain a first order differential equation by describing the rate of change of capacitor voltage

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L$$

Kirchhoff's voltage low for the right-hand loop provides the equation describing the rate of change of inducator current as

$$L\frac{di_L}{dt} = -R\,i_L + v_c$$

The output of the system is represented by the linear algebraic equation

$$v_0 = R\,i_L(t)$$

We can write the equations as a set of two first order differential equations in terms of the state variables x_1 [$v_C(t)$] and x_2 [$i_L(t)$] as follows:

$$C\frac{dv_c}{dt} = u(t) - i_L \quad \Longrightarrow \quad \frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$L\frac{di_L}{dt} = -R\,i_L + v_c \quad \Longrightarrow \quad \frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

The output signal is then $y_1(t) = v_0(t) = R\,x_2$

Utilizing the first-order differential equations and the initial conditions of the network represented by [$x_1(t_0)$ $x_2(t_0)$], we can determine the system's future and its output.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For the RLC circuit, we might choose the set of state variables as the two voltages, $v_C(t)$ and $v_L(t)$.

In an actual system, there are several choices of a set of state variables that specify the *energy stored in a system* and therefore adequately describe the dynamics of the system.

The state variables of a system characterize the dynamic behavior of a system. The engineer's interest is primarily in physical, where the variables are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.

The State Differential Equation:

The state of a system is described by the set of first-order differential equations written in terms of the state variables [x_1 x_2 ... x_n]. These first-order differential equations can be written in general form as

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + b_{11}u_1 + \cdots b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n + b_{21}u_1 + \cdots b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n + b_{n1}u_1 + \cdots b_{nm}u_m$$

Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

n: number of state variables, m: number of inputs.

The column matrix consisting of the state variables is called the state vector and is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The vector of input signals is defined as u. Then the system can be represented by the compact notation of the state differential equation as

$$\dot{x} = A x + B u$$

This differential equation is also commonly called the state equation. The matrix A is an nxn square matrix, and B is an nxm matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation

$$y = C x + D u$$

Where y is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.

We can write the state variable differential equation for the RLC circuit as

$$\dot{x} = \begin{bmatrix} 0 & -\dfrac{1}{C} \\ \dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} x + \begin{bmatrix} \dfrac{1}{C} \\ 0 \end{bmatrix} u(t)$$

and the output as

$$y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax + bu$$

Where x(t) and u(t) are scalar functions of time. We expect an exponential solution of the form $e^{at.}$ Taking the Laplace transform of both sides, we have

$$s\,X(s) - x_0 = a\,X(s) + b\,U(s)$$

therefore,

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}\,U(s)$$

The inverse Laplace transform of X(s) results in the solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}b\,u(\tau)\,d\tau$$

We expect the solution of the state differential equation to be similar to x(t) and to be of differential form. The matrix exponential function is defined as

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

which converges for all finite t and any A. Then the solution of the state differential equation is found to be

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

where we note that $[sI\text{-}A]^{-1}=\phi(s)$, which is the Laplace transform of $\phi(t)=e^{At}$. The matrix exponential function $\phi(t)$ describes the unforced response of the system and is called the fundamental or state transition matrix.

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$

THE TRANSFER FUNCTION FROM THE STATE EQUATION

The transfer function of a single input-single output (SISO) system can be obtained from the state variable equations.

$$\dot{x} = A\,x + B\,u$$
$$y = C\,x$$

where y is the single output and u is the single input. The Laplace transform of the equations

$$sX(s) = AX(s) + B\,U(s)$$
$$Y(s) = CX(s)$$

where B is an nx1 matrix, since u is a single input. We do not include initial conditions, since we seek the transfer function. Reordering the equation

$$[sI - A]X(s) = BU(s)$$

$$X(s) = [sI - A]^{-1}BU(s) = \phi(s)BU(s)$$

$$Y(s) = C\phi(s)BU(s)$$

Therefore, the transfer function G(s)=Y(s)/U(s) is

$$G(s) = C\phi(s)B$$

Example:

Determine the transfer function G(s)=Y(s)/U(s) for the RLC circuit as described by the state differential function

$$\dot{x} = \begin{bmatrix} 0 & -\dfrac{1}{C} \\ \dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} x + \begin{bmatrix} \dfrac{1}{C} \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

$$[sI - A] = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{bmatrix}$$

$$\phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

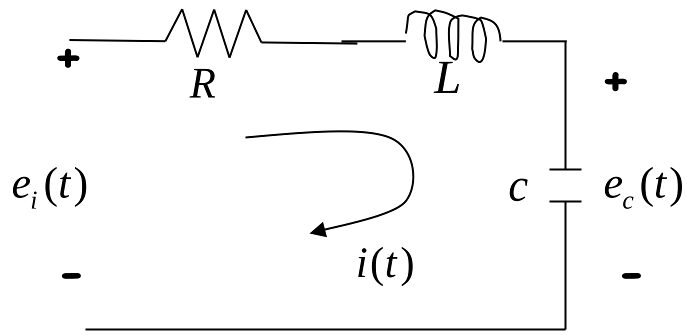
$$\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

Then the transfer function is

$$G(s) = [0 \quad R] \begin{bmatrix} \dfrac{s + \frac{R}{L}}{\Delta(s)} & -\dfrac{1}{C\Delta(s)} \\ \dfrac{1}{L\Delta(s)} & \dfrac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$G(s) = \frac{R / LC}{\Delta(s)} = \frac{R / LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Remark : ***the choice of states is not unique.***



$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{c}\int_0^t i(t)dt = e_i(t)$$

let

$$x_1(t) = i(t)$$
$$x_2(t) = \int i(t)dt$$
$$y(t) = i(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} e_i(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

exist a mapping

let

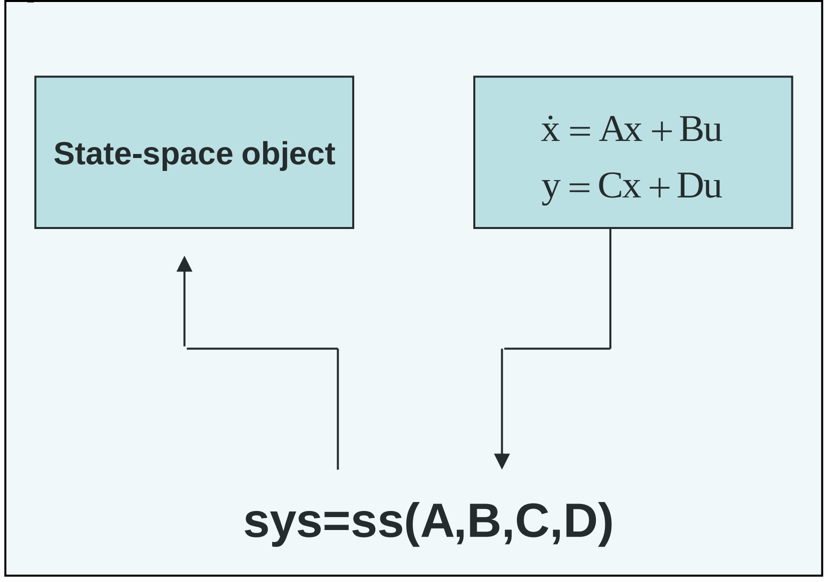
$$\hat{x}_1(t) = i(t)$$
$$\hat{x}_2(t) = e_c(t)$$
$$y(t) = i(t)$$

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{R}{L} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} e_i(t)$$

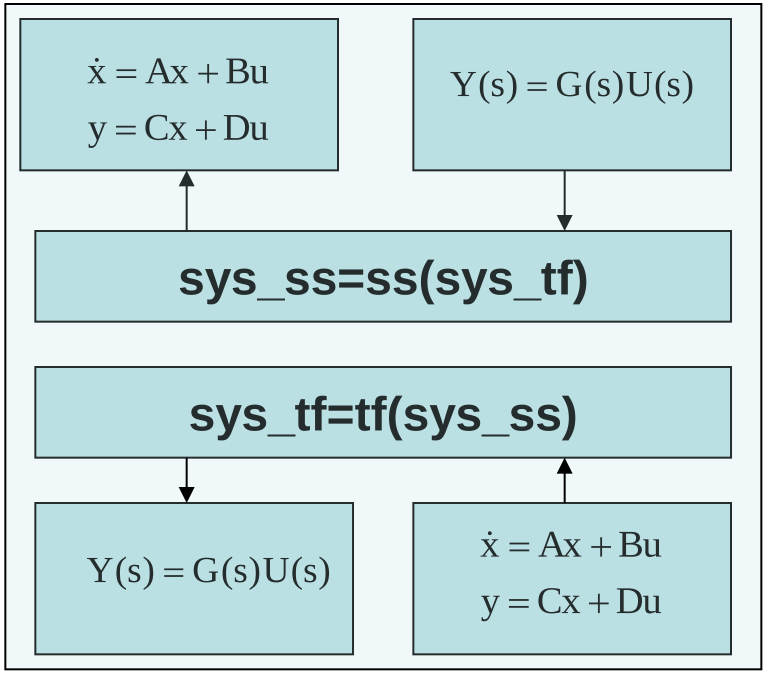
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

ANALYSIS OF STATE VARIABLE MODELS USING MATLAB

Given a transfer function, we can obtain an equivalent state-space representation and vice versa. The function tf can be used to convert a state-space representation to a transfer function representation; the function ss can be used to convert a transfer function representation to a state-space representation. The functions are shown in Figure 4, where sys_tf represents a transfer function model and sys_ss is a state space representation.



The ss function



Linear system model conversion

For instance, consider the third-order system

$$G(s)=\frac{Y(s)}{R(s)}=\frac{2s^2+8s+6}{s^3+8s^2+16s+6}$$

We can obtain a state-space representation using the ss function. The state-space representation of the system given by G(s) is

<u>Matlab code</u>

```
num=[2 8 6];den=[1 8 16 6];
sys_tf=tf(num,den)
sys_ss=ss(sys_tf)
```

Answer

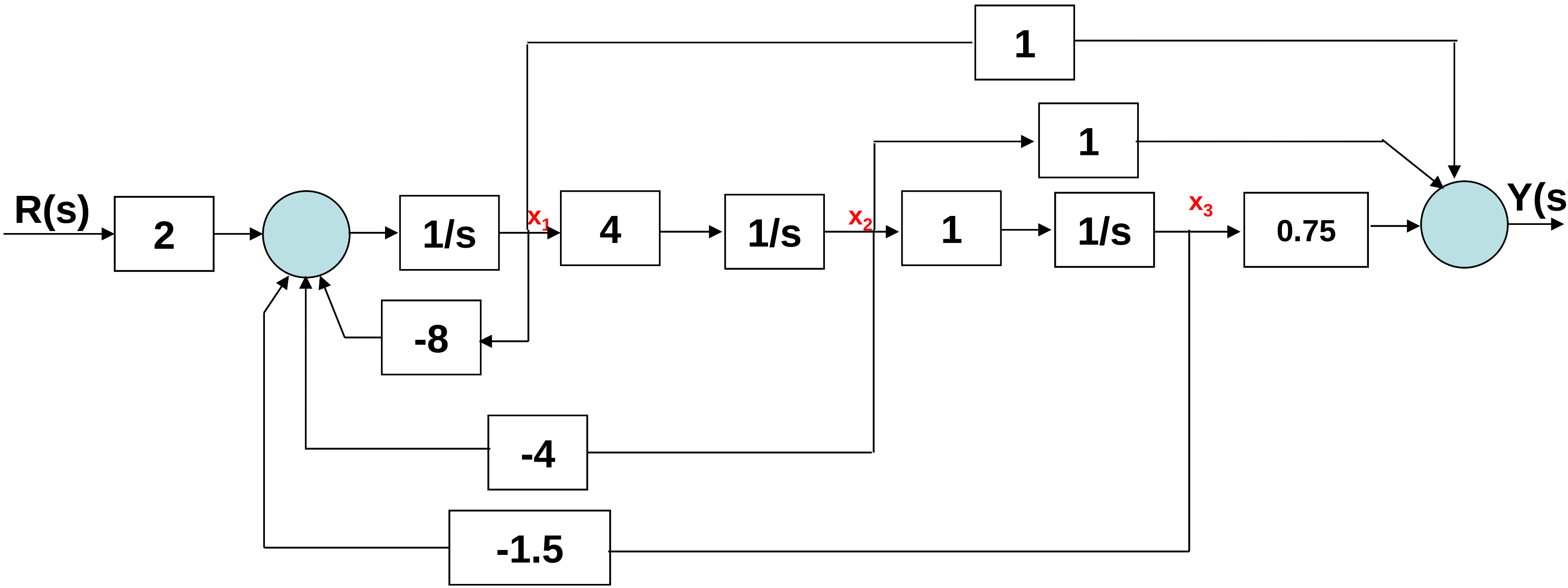
```
Transfer function:
  2 s^2 + 8 s + 6
----------------------
s^3 + 8 s^2 + 16 s + 6


a =
        x1    x2    x3
  x1    -8    -4  -1.5
  x2     4     0     0
  x3     0     1     0

b =
      u1
  x1   2
  x2   0
  x3   0

c =
        x1    x2    x3
  y1     1     1  0.75

d =
      u1
  y1   0

Continuous-time model.
```

$$A=\begin{bmatrix}-8 & -4 & -1.5\\ 4 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ B=\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 1 & 0.75\end{bmatrix}\text{and } D=\begin{bmatrix}0\end{bmatrix}$$

$$A = \begin{bmatrix} -8 & -4 & -1.5 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0.75 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 \end{bmatrix}$$



Block diagram with x_1 defined as the leftmost state variable.

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$x(t) = \phi(t) x(0) + \int_0^t \phi(t-\tau) B u(\tau) d\tau$$

We can use the function expm to compute the transition matrix for a given time. The expm(A) function computes the matrix exponential. By contrast the exp(A) function calculates $e^{a_{ij}}$ for each of the elements $a_{ij} \epsilon A$.

For the RLC network, the state-space representation is given as:

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{and } D = \begin{bmatrix} 0 \end{bmatrix}$$

The initial conditions are $x_1(0)=x_2(0)=1$ and the input u(t)=0. At t=0.2, the state transition matrix is calculated as

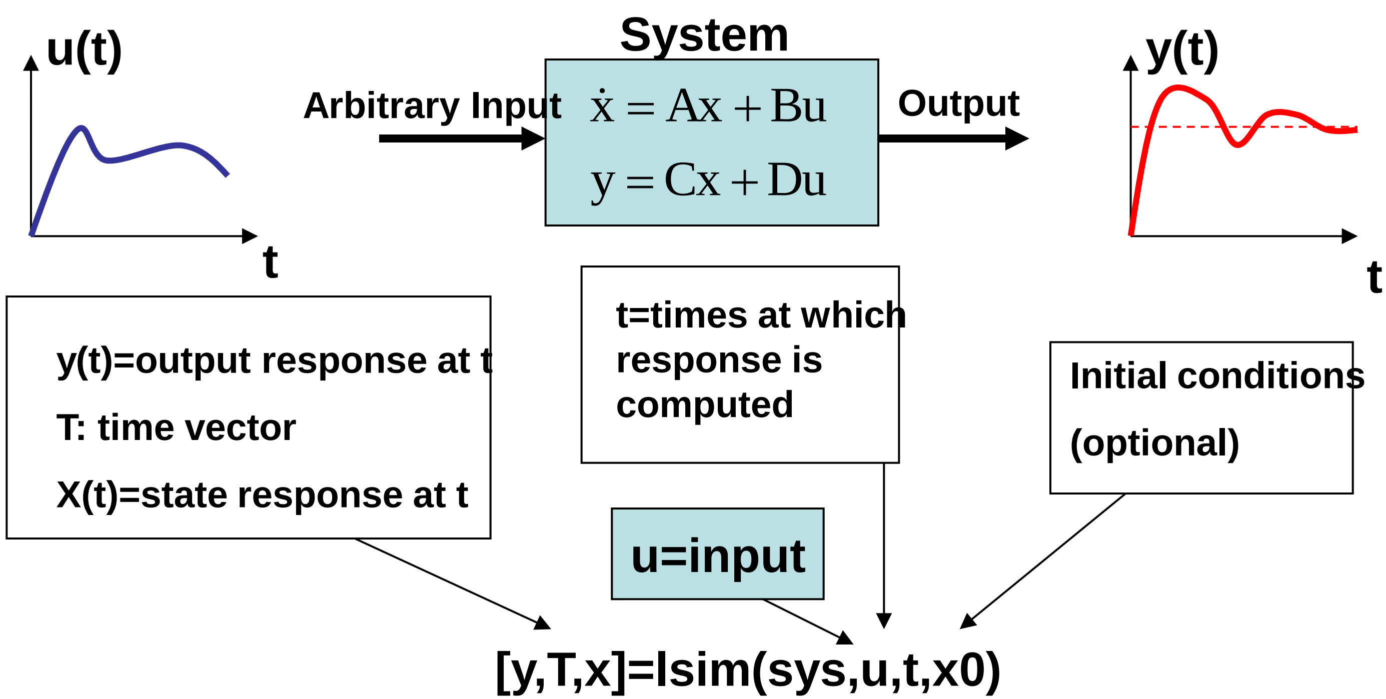
>>A=[0 -2;1 -3], dt=0.2; Phi=expm(A*dt)

```
Phi =

    0.9671   -0.2968
    0.1484    0.5219
```

The state at t=0.2 is predicted by the state transition method to be

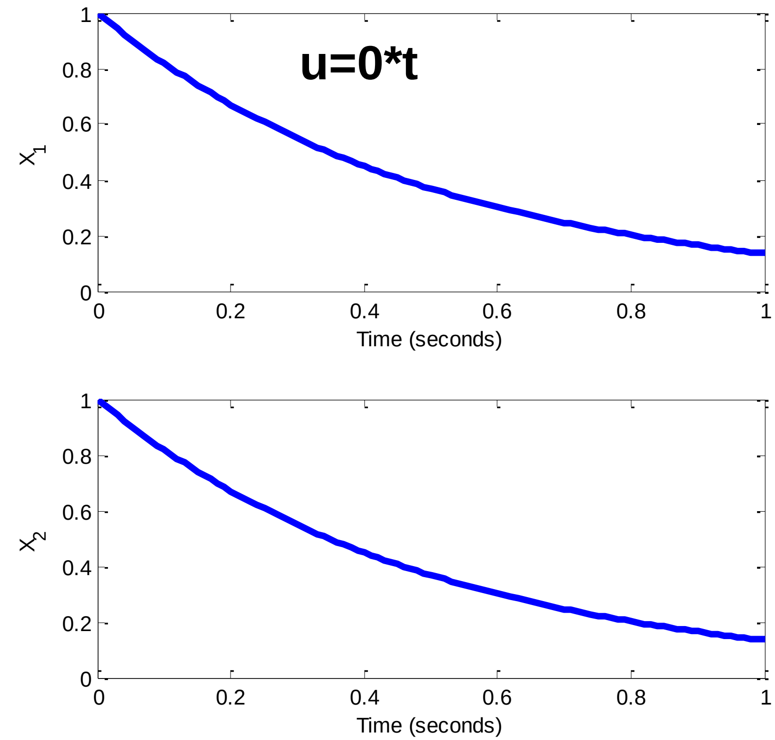
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{t=0.2} = \begin{bmatrix} 0.9671 & -0.2968 \\ 0.1484 & 0.5219 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{t=0} = \begin{bmatrix} 0.6703 \\ 0.6703 \end{bmatrix}$$

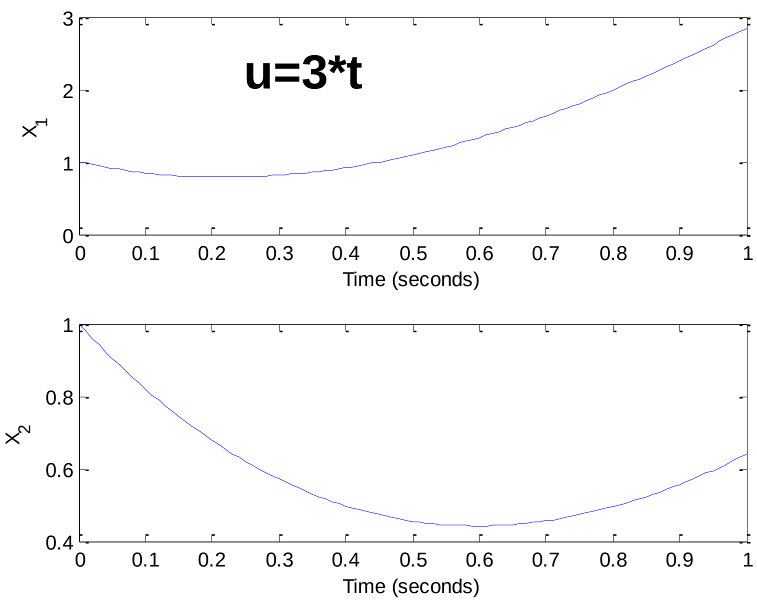
The time response of a system can also be obtained by using lsim function. The lsim function can accept as input nonzero initial conditions as well as an input function. Using lsim function, we can calculate the response for the RLC network as shown below.

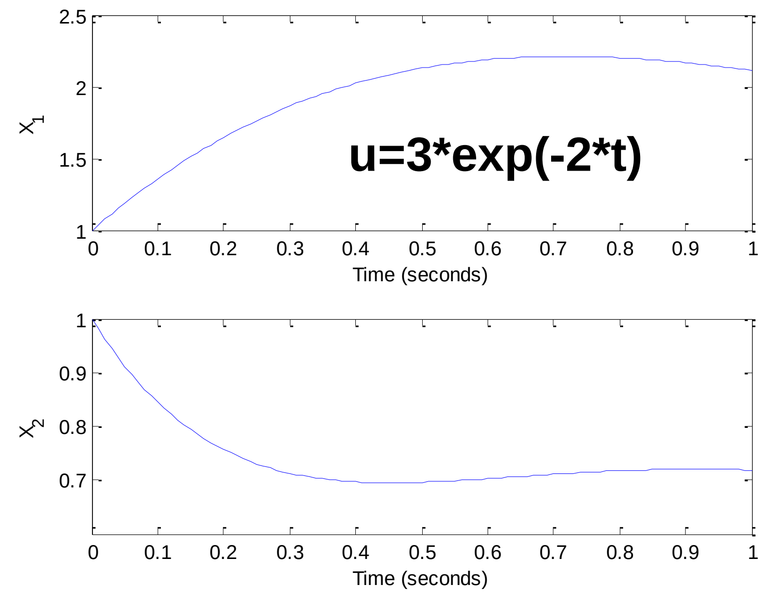


Matlab code

```
clc;clear
A=[0 -2;1 -3];B=[2;0];C=[1 0];D=[0];
sys=ss(A,B,C,D)   %state-space model
x0=[1 1]; %initial conditions
t=[0:0.01:1];
u=0*t;  %zero input
[y,T,x]=lsim(sys,u,t,x0);
subplot(211),plot(T,x(:,1))
xlabel('Time (seconds)'),ylabel('X_1')
subplot(212),plot(T,x(:,2))
xlabel('Time (seconds)'),ylabel('X_2')
```







With Our Best Wishes
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Course Staff